Optimization of Network Robustness to Waves of Targeted and Random Attacks

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We study the robustness of complex networks to multiple waves of simultaneous (i) targeted attacks in which the highest degree nodes are removed and (ii) random attacks (or failures) in which fractions p_t and p_r respectively of the nodes are removed until the network collapses. We find that the network design which optimizes network robustness has a bimodal degree distribution, with a fraction r of the nodes having degree $k_2 = (\langle k\rangle - 1 + r)/r$ and the remainder of the nodes having degree $k_1 = 1$, where $\langle k\rangle$ is the average degree of all the nodes. We find that the optimal value of r is of the order of p_t/p_r for $p_t/p_r \ll 1$.

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Recently, there has been much interest in the resilience of real-world networks to random attacks or to attacks targeted on the highest degree nodes [1, 2, 3, 4, 5, 6, 7, 8]. Many real-world networks are robust to random attacks but vulnerable to targeted attacks. It is important to understand how to design networks which are optimally robust against both types of attacks, with examples being terrorist attacks on physical networks and attacks by hackers on computer networks. Studies to date [7, 8] have considered only the case in which there was only one type of attack on a given network — that is, the network was subject to either a random attack or to a targeted attack but not subject to different types of attack simultaneously.

A more realistic scenario is one in which a network is subjected to simultaneous targeted and random attacks. This scenario can be modeled as a sequence of "waves" of targeted and random attacks which remove fractions p_t and p_r of the original nodes, respectively. The ratio p_t/p_r is kept constant while the individual fractions p_t and p_r approach zero. After some time (after m waves of random and targeted attacks) the network will become disconnected; at this point a fraction $f_c = m(p_t + p_r)$ of the nodes have been removed. This f_c characterizes the network robustness. The larger f_c, the more robust the network is. We propose in this Letter a mathematical approach to study such simultaneous attacks and find the optimal network design one which maximizes f_c In our optimization analysis, we compare the robustness of networks which have the same "cost" of construction and maintenance, where we define cost to be proportional to the average degree $\langle k\rangle$ of all the nodes in the network.

We study mainly two types of random networks:

(i) *Scale-free networks.* Many real world computer, social, biological and other types of networks have been found to be scale free, i.e., they exhibit degree distributions of the form $P(k) \sim k^{-\lambda}$ [9, 10, 11, 12, 13, 14, 15, 16, 17]. For large scale-free networks with exponent λ less than 3, for random attacks essentially all nodes must be removed for the network to become disconnected [3, 4]. On the other hand, because the scale free distribution has a long power-law tail (i.e. hubs with large degree), the scale-free networks are very vulnerable with respect to targeted attack.

(ii) *Networks with bimodal degree distributions.* For resilience to single random or single targeted attacks, certain bimodal distributions are superior to any other network [7, 8]. Here we ask if these networks are also most resilient to multiple waves of both random and targeted attacks.

We present the following argument that suggests that the degree distribution which optimizes f_c is a bimodal distribution in which a fraction r of the nodes has degree

$$k_2 = \frac{\langle k\rangle - 1 + r}{r} \tag{1}$$

and the remainder has degree $k_1 = 1$ and we show that r is of the order of p_t/p_r. To optimize against random removal, we maximize the quantity $\kappa \equiv \langle k^2\rangle/\langle k\rangle$, since for random removal the threshold is [3]

$$f_c^{\mathrm{rand}} = 1 - \frac{1}{\kappa - 1}. \tag{2}$$

Since we keep $\langle k\rangle$ fixed, κ is just the variance of the degree distribution and is maximized for a bimodal distribution in which the lower degree nodes have the smallest possible degree $k_1 = 1$ and the higher degree nodes have the highest possible degree consistent with keeping $\langle k\rangle$ fixed, $k_2 = (\langle k\rangle - 1 + r)/r$. Thus, k_2 is maximized when r assumes its smallest possible value, $r = 1/N$. On the other hand, if all of the high degree nodes are removed by targeted attacks, the network will be very vulnerable to random attack. So we want to delay as long as possible the situation in which all of the high degree nodes

are removed by targeted attacks—which argues for not choosing r as small as possible but choosing r such that some high connectivity nodes remain as long as there are some low connectivity nodes. Such a condition is achieved when r is of the order of p_t/p_r.

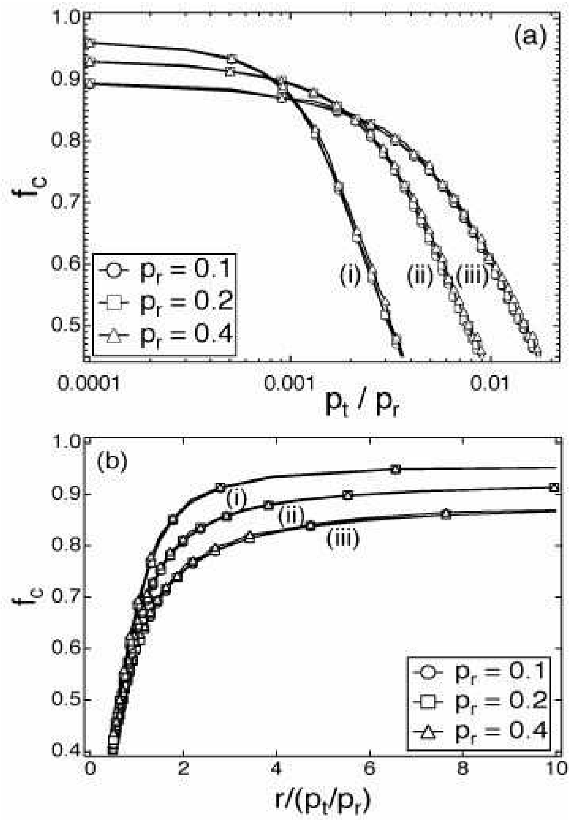

FIG. 1: (a) The threshold f_c of three bimodal networks with $\langle k \rangle = 3$, with (i) $r = 2 \times 10^{-3}$ and $k_2 = 200$, (ii) $r = 5 \times 10^{-3}$ and $k_2 = 90$, and (iii) $r = 10^{-2}$ and $k_2 = 50$. The results are plotted as a function of the ratio p_t/p_r for three fixed values of p_r. These plots show that the values of the threshold are dependent only on the ratio p_t/p_r and independent of the value of p_r itself. (b) Scaled plot of the data in (a). The data show that the plots collapse in the region where $r/(p_t/p_r) \lesssim 1$.

The method we employ for determining the threshold makes use of the following: the general condition for a random network to be globally connected is [3, 5, 6]

$$\kappa = \frac{\langle k^2 \rangle}{\langle k \rangle} \geq 2. \tag{3}$$

Random removal of a fraction p_r of nodes from a network with degree distribution $P_0(k)$ results in a new degree distribution [3]

$$P(k) = \sum_{k_0=k}^{K} P_0(k) \binom{k_0}{k} (1-p_r)^k \, p_r^{k_0-k}, \tag{4}$$

where K is the upper cutoff of the degree distribution. Targeted removal of a fraction p_t of the highest degree nodes reduces the value of upper cutoff K to $\tilde{K}$, which is implicitly determined by the equation

$$p_t = \sum_{k=\tilde{K}}^{K} P_0(k). \tag{5}$$

The removal of high degree nodes causes another effect. Since the links that lead to removed nodes are eliminated, the degree distribution also changes. This effect is equivalent to the random removal of a fraction of $\tilde{p}$ nodes where

$$\tilde{p} = \frac{\sum_{k=\tilde{K}}^{K} k P_0(k)}{\langle k \rangle_0}. \tag{6}$$

The average $\langle k \rangle_0$ is taken over the degree distribution before the removal of nodes [5]. Equation (4) with p_r replaced by $\tilde{p}$ can then be used to calculate the effect of the link removal. Starting with a certain initial degree distribution, we recursively calculate $P(k)$ alternating between random and targeted attack using Eqs. (4), (5), and (6), and calculate κ after each wave of attacks. When $\kappa < 2$ global connectivity is lost and $f_c = m(p_r + p_t)$ where m is the number of waves of attacks performed.

We begin our study by first establishing numerically that, for small values of p_t, p_r and p_t/p_r, the threshold f_c depends only on p_t/p_r. In Fig. 1(a), we plot the threshold f_c of a network with a bimodal degree distribution with $\langle k \rangle = 3$ for various values of p_r and r as a function of the ratio p_t/p_r. The collapse of the plots with the same r but different p_r shows that the values of threshold are essentially independent of the value of p_r itself but depend only on the ratio p_t/p_r [23].

In Fig. 1(b) we plot f_c against the scaled variable $r/(p_t/p_r)$. We see that the plots for different values of r collapse, indicating that only the scaled variable $r/(p_t/p_r)$ is relevant [24].

Next we study the dependence of f_c on k_2. As seen in Fig. 2, as expected the maximum values of f_c for various values of p_t/p_r are obtained when k_2 is maximum (i.e., when $k_1 = 1$, see Eq. (1).).

We are now in a position to determine the value of r which optimizes f_c, r_{opt}. In Fig. 3, we plot f_c as a function of the scaled parameter $r/(p_t/p_r)$ with k_2 set to the maximum value possible for each value of r. We note that there is a transition at a well-defined value of $r/(p_t/p_r)$ at which f_c increases rapidly to a shallow maximum f_c^{opt} at $r_{\text{opt}}/(p_t/p_r) \approx 1.7$. This value of $r_{\text{opt}}/(p_t/p_r)$ is valid for $p_t/p_r \ll 1$. In order to determine $r_{\text{opt}}/(p_t/p_r)$ over a wider range, we make extensive numerical calculations for $10^{-3} < p_t/p_r < 0.1$. For each value of p_t/p_r, we calculate the value $r_{\text{opt}}/(p_t/p_r)$ where f_c takes its maximum value and find

$$\frac{r_{\text{opt}}}{p_t/p_r} \approx 1.7 - 5.6 \left(\frac{p_t}{p_r}\right) + O\left(\frac{p_t}{p_r}\right)^2 \tag{7}$$

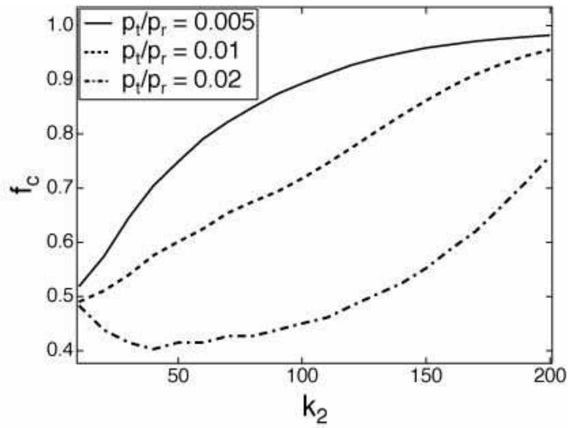

FIG. 2: The threshold f_c versus k_2 for a bimodal network with $\langle k \rangle = 3$ and $r = 10^{-2}$ for three values of p_t/p_r. The value of p_r is fixed at 0.02. For each value of p_t/p_r, the thresholds take their maximum values at the maximum k_2 (obtained when $k_1 = 1$).

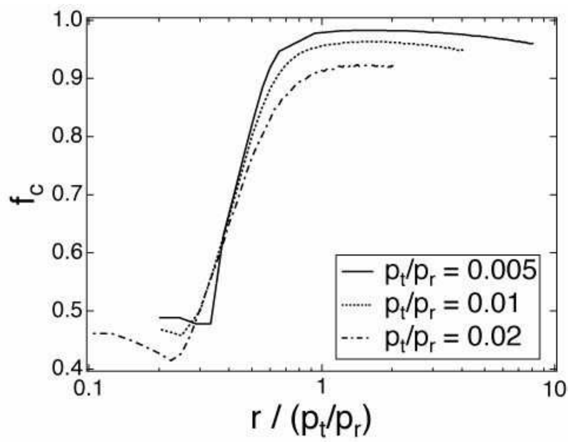

FIG. 3: The threshold f_c versus the scaled parameter $r/(p_t/p_r)$ for a bimodal network with $\langle k \rangle = 3$ and k_2 maximum (i.e., $k_1 = 1$).

within the range of our calculation. For larger values of p_t/p_r, $r_{\rm opt} = 1$ and from Eq. (1) all nodes have degree $\langle k \rangle$. In Fig. 4, we plot the values of the optimal threshold $f_c^{\rm opt}$ by a thick solid curve.

In Fig. 4 we also plot the values of the threshold f_c for the same bimodal network but we fix r independent of p_t/p_r. We see that these configurations are not significantly less robust than the optimal configuration. Thus, even if we do not know the ratio p_t/p_r exactly we can design networks which will be relatively robust. For example, the bimodal network with $r = 0.03$ is relatively robust for $p_t/p_r \lesssim 0.1$ and the bimodal network with $r = 0.09$ is robust for $p_t/p_r \lesssim 1$. Also plotted in Fig. 4 is the optimal scale-free network with $\langle k \rangle = 3$. We see that the optimal bimodal network is more robust than the optimal scale-free network and we can even pick a

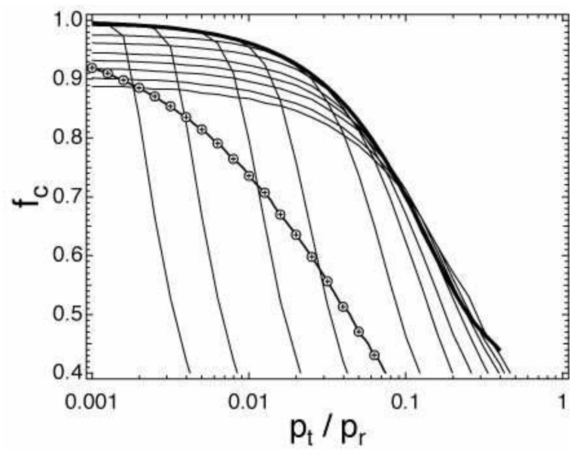

FIG. 4: The threshold f_c versus p_t/p_r. The topmost (thickest) curve is for a bimodal network with $\langle k \rangle = 3$ with $k_1 = 1$ and with r optimized by Eq. (7) for each value of p_t/p_r. The values of the threshold for the same bimodal network with $k_1 = 1$ when we fix r independent of p_t/p_r are plotted in thin curves. The values of r are $r = 0.001, 0.002, 0.005, 0.01, 0.03, 0.05, 0.07, 0.09, 0.11, 0.13,$ and 0.15, from left to right. The curve marked with crossed circles ($\oplus$) is a plot of the threshold values for a scale-free network with $\langle k \rangle = 3$, $N = 10^4$, and with exponent chosen for each p_t/p_r to optimize the threshold. Note that the thresholds for bimodal networks with $0.03 \lesssim r \lesssim 0.09$ are always more robust than the optimized scale-free network.

configuration with fixed r (e.g. $r = 0.03$) which is more robust than the optimal scale-free network in most ranges of p_t/p_r.

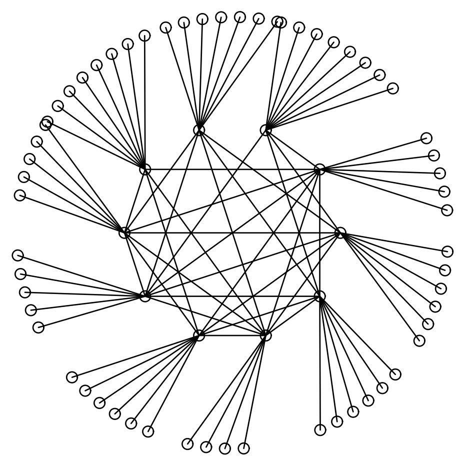

FIG. 5: Realization of bimodal network with $N = 100$ nodes, $\langle k \rangle = 2.1$ and $r = 0.1$, so there are $rN = 10$ "hub" nodes of degree 12, as found from Eq. (1).

In Fig. 5 we show a typical optimal realization of a bimodal network. The network of $N = 100$ nodes consists of rN nodes with $k = k_2$ ("hubs") which are highly connected among themselves; the nodes of single degree are each connected to one of these hubs. We note that

while the hubs are highly connected among themselves, they do not form a complete graph — every hub is not connected to every other hub. For larger N, the fraction of hubs to which a given hub connects decreases but the robustness of the network is unchanged.

In summary, we have provided a qualitative argument and numerical results which indicate that the most robust network to multiple waves of targeted and random attacks has a bimodal degree distribution with a fraction r of the nodes having degree $k_2 = (\langle k \rangle - 1 + r)/r$ and the remainder of the nodes having degree 1. The optimal value of r is approximately $1.7\,(p_t/p_r)$ for $p_t/p_r \ll 1$. For larger values of p_t/p_r, the optimal value of r is 1 and all nodes have degree $\langle k \rangle$. Even if p_t/p_r is not known exactly, a value of r can be chosen which maximizes the network robustness over a wide range of values of p_t/p_r, as seen in Fig. 4.

We note that while the optimal distribution found here and that found in Ref. [8] are both bimodal, the values of the parameters characterizing these distributions are different. As found in Ref. [8] the network with optimal resilience to either random or targeted attack has $r = 1/N$ and $k_2 \sim r^{-2/3}$. Finally, we note that it is possible to prove analytically that for the case in which a single targeted attack followed by a single random attack results in the network becoming disconnected, the optimal distribution is also bimodal with $k_1 = 1$, $k_2 = (\langle k \rangle - 1 + r)/r$ and r of the order of p_t/p_r [25] — supporting the results found here for multiple waves of attacks.

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